Tuesday, March 17, 2015 6:39 PM

Conserved Quantities and Spacetime Symmetries

You should be familiar with the following argument from nonrelativistic mechanics:

Euler-Lagrange equations $\frac{d}{dt}\Big(\underbrace{\frac{\partial L}{\partial \dot{x}^i}}_{\equiv p^i}\Big) - \frac{\partial L}{\partial x^i} = 0 \Rightarrow \frac{dp^i}{dt} = \frac{\partial L}{\partial x^i} \Rightarrow$ if L does not depend on x^i

$\equiv p^i$ momentum conjugate to coordinate x^i then p^i is constant

This is a simple version of a more powerful argument by Emmy Noether that also applys to field theories and internal as well as spacetime symmetries. Generically: A continuous symmetry of an action gives rise to a conserved current.

Symmetries and conserved quantities are incredibly useful tools for solving equations of motion (think W-E, I-M).

We won't work with a Lagrangian formulation of GR, but if we restrict to spacetime symmetries (isometries) then we can take a slightly different approach.

We will carelessly interchange conserved w/ constant, though a distinction should be made!

Normally we make statements of the form "a symmetry in x means that $\frac{dp^x}{dt} = 0$" which is highly coordinate dependent. We can do something similar in GR.

We know that 4-momenta are given by $p^\mu = m U^\mu = m \frac{dx^\mu}{d\tau}$ for massive particles.

The geodesic equation can be written:

$$\frac{dx^\nu}{d\tau} \nabla_\nu \frac{dx^\mu}{d\tau} = 0 = p^\nu \nabla_\nu p^\mu \quad \left(= m \frac{Dp^\mu}{d\tau}\right)$$

Let's hit both sides with $g_{\alpha\mu}$ and use $\nabla_\nu g_{\alpha\mu} = 0$ (metric compatibility):

$$\begin{aligned}
0 &= p^\nu \nabla_\nu p_\alpha \\
&= p^\nu (\partial_\nu p_\alpha - \Gamma^\lambda_{\nu\alpha} p_\lambda) \\
&= \underbrace{p^\nu \partial_\nu p_\alpha}_{m \frac{dx^\nu}{d\tau} \partial_\nu p_\alpha} - \underbrace{p^\nu \Gamma^\lambda{}_{\nu\alpha} p_\lambda}_{p^\nu \frac{1}{2} g^{\lambda\beta} (\partial_\nu g_{\alpha\beta} + \partial_\alpha g_{\beta\nu} - \partial_\beta g_{\nu\alpha}) p_\lambda} \\
&= m \frac{dp_\alpha}{d\tau} - \frac{1}{2} (\partial_\nu \underbrace{g_{\alpha\beta}}_{= g_{\beta\alpha}} + \partial_\alpha g_{\beta\nu} - \partial_\beta g_{\nu\alpha}) p^\nu p^\beta \\
&= m \frac{dp_\alpha}{d\tau} - \frac{1}{2} \underbrace{(\partial_\nu g_{\beta\alpha} - \partial_\beta g_{\nu\alpha})}_{\text{antisymmetric under } \nu \leftrightarrow \beta} \underbrace{p^\nu p^\beta}_{\text{symmetric under } \nu \leftrightarrow \beta} - \frac{1}{2} (\partial_\alpha g_{\beta\nu}) p^\nu p^\beta
\end{aligned}$$

$$0 = m \frac{dp_\alpha}{d\tau} - \frac{1}{2} (\partial_\alpha g_{\beta\nu}) p^\nu p^\beta$$

Hence we find that $\frac{dp_\alpha}{d\tau} = 0$ if $\partial_\alpha g_{\beta\nu} = 0$ (or if $g_{\beta\nu}$ is independent of x^α).

This is a conservation result similar to p^x is conserved if L is independent of x.

Tuesday, March 17, 2015 8:12 PM

But we can do better! So far our discussion has been very coordinate dependent. We can say that for a given choice of coordinates, if the metric is independent of one coordinate then there is a corresponding conserved momentum. But we would like a coordinate independent way of identifying symmetries (and conserved quantities).

Suppose that p_{σ^*} is a conserved momentum component, i.e. $\frac{dp^{\sigma^*}}{d\tau} = 0$.

We will introduce a vector K^μ such that $K^\mu p_\mu = p^{\sigma^*} = K_\mu p^\mu$.

Then: $$\frac{dp^{\sigma^*}}{d\tau} = 0 = \frac{d(K_\mu p^\mu)}{d\tau} = \frac{dx^\nu}{d\tau}\partial_\nu (K_\mu p^\mu) = \frac{dx^\nu}{d\tau}\nabla_\nu (K_\mu p^\mu)$$

same when acting on scalars!

Multiplying by m: $$0 = p^\nu p^\mu \nabla_\nu K_\mu + K_\mu \underbrace{p^\nu \nabla_\nu p^\mu}_{=0 \text{ by geodesic equation}}$$

$$0 = p^\nu p^\mu \nabla_{(\nu} K_{\mu)} + \underbrace{p^\nu p^\mu \nabla_{[\nu} K_{\mu]}}_{=0 \text{ since S}\cdot\text{A}}$$

Then we find that if $\nabla_{(\nu}K_{\mu)} = 0$ then K_μ is a symmetry of the geometry and $K_\mu p^\mu$ is a conserved quantity!

dual Killing vector

Killing equation

3.9

Monday, March 23, 2015 7:38 AM

How many solutions to $\nabla_{(\mu}K_{\nu)}=0$ should we expect?
In general this is hard to know in advance, but for maximally symmetric spaces we have a simple answer.

Recall that locally any manifold looks like $\mathbb{R}^n$ or $\mathbb{M}^n$. These allow:

n translations
$\frac{1}{2}n(n-1)$ rotations ($\mathbb{R}^n$) or Lorentz trans. ($\mathbb{M}^n$)
} Together these form the Euclidean or Poincare groups.

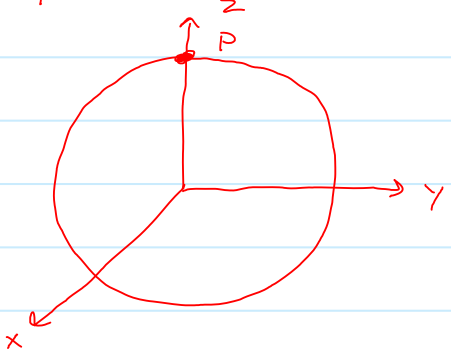
In total: $n+\frac{1}{2}n^2-\frac{1}{2}n=\underbrace{\frac{1}{2}n(n+1)}_{10 \text{ in } 4D}$ local symmetries

If all of the local symmetries are also valid globally then the space (time) is called maximally symmetric and we should expect $\frac{1}{2}n(n+1)$ independent solutions to $\nabla_{(\mu}K_{\nu)}=0$.

Monday, March 23, 2015 7:49 AM

You might guess that only $\mathbb{R}^n$ or $\mathbb{M}^n$ themselves are maximally symmetric, but that's not quite the case.

S^2



For S^2 we know (via its embedding into $\mathbb{R}^3$) that it is symmetric under rotations around x, y, z. So we have $3 = \frac{1}{2} d(d+1)$ symmetries.

To see the local trans. + rotation breakdown, consider the north pole (P).

Then: R_z is the single $\frac{1}{2} d(d-1) = 1$ rotation

R_y, R_x are the d translations

S^2 is maximally symmetric, even though it definitely isn't $\mathbb{R}^2$ (it's curved!)

Maximally symmetric spaces do not have to be flat, but their curvature does take a simple form.

Due to the translation invariance, if we know $R^{\nu}{}_{\rho\mu\nu}$ at any point, it must have the same value at any other point, i.e. $R^{\lambda}{}_{\rho\mu\nu}(x^{\mu}) = R^{\lambda}{}_{\rho\mu\nu}$ = constant!

In fact, just knowing the Ricci scalar and the metric one can show:
For maximally symmetric spaces: $R_{\lambda\rho\mu\nu} = \frac{R}{n(n-1)} \underbrace{(g_{\lambda\mu} g_{\rho\nu} - g_{\lambda\nu} g_{\rho\mu})}$ (no derivatives!)

Note: antisymm $\begin{cases} \lambda \leftrightarrow \rho \\ \mu \leftrightarrow \nu \end{cases}$

symm $\lambda\rho \leftrightarrow \mu\nu$

A catalog of maximally symmetric spaces:

	Euclidean	Lorentzian
$R=0$	$\mathbb{R}^n$, $\mathbb{T}^n$ (Euclidean, tori)	$\mathbb{M}^n$, $\mathbb{M}^n \times \mathbb{T}^m$ (Minkowski, Minkowski x tori)
$R>0$	S^n (spheres)	dS^n (de Sitter)
$R<0$	H^n (hyperbolic)	AdS^n (anti-de Sitter)

Let's go through all the gory details for S^2.

S^2 w/ (θ,ϕ) : $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$ $\quad g_{\mu\nu} = \begin{pmatrix} 1 & \\ & \sin^2\theta \end{pmatrix}$ $\quad g^{\mu\nu}\begin{pmatrix} 1 & \\ & 1/\sin^2\theta \end{pmatrix}$

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta \qquad \Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta \qquad \text{all other } \Gamma\text{'s} = 0$$

We expect 3 ind. solutions to $\nabla_{(\mu} K_{\nu)} = 0 = \partial_\mu K_\nu - \Gamma^\lambda_{\mu\nu} K_\lambda + \partial_\nu K_\mu - \Gamma^\lambda_{\nu\mu} K_\lambda$

$$= \partial_\mu K_\nu + \partial_\nu K_\mu - 2\Gamma^\lambda_{\mu\nu} K_\lambda$$

There are 3 distinct equations here:

$\mu=\nu=\theta \qquad 0 = 2\partial_\theta K_\theta - 2\Gamma^\lambda_{\theta\theta} K_\lambda = \partial_\theta K_\theta$

$\mu=\nu=\phi \qquad 0 = 2\partial_\phi K_\phi - 2\Gamma^\lambda_{\phi\phi} K_\lambda = \partial_\phi K_\phi + \sin\theta\cos\theta\, K_\theta$

$\mu=\theta, \nu=\phi \qquad 0 = \partial_\theta K_\phi + \partial_\phi K_\theta - 2\Gamma^\lambda_{\theta\phi} K_\lambda = \partial_\theta K_\phi + \partial_\phi K_\theta - 2\cot\theta\, K_\phi$

3 ind. solutions (each solution satisfies <u>all</u> 3 equations above!) are:

$K^1_\mu = (0, \sin^2\theta)$

$K^2_\mu = (\sin\phi, \tfrac{1}{2}\cos\phi\sin(2\theta))$

$K^3_\mu = (\cos\phi, -\tfrac{1}{2}\sin\phi\sin(2\theta))$

Each of these corresponds to a conserved quantity. If $p^\mu = (p^\theta, p^\phi)$ then:

$$\left.\begin{aligned} K^1_\mu p^\mu &= \sin^2\theta\, p^\phi \\ K^2_\mu p^\mu &= \sin\phi\, p^\theta + \tfrac{1}{2}\cos\phi\sin(2\theta)\, p^\phi \\ K^3_\mu p^\mu &= \cos\phi\, p^\theta - \tfrac{1}{2}\sin\phi\sin(2\theta)\, p^\phi \end{aligned}\right\} \text{All are conserved, i.e. } \frac{d(K_\mu p^\mu)}{d\tau} = 0 \left\{\begin{aligned} &= L_z \\ &\simeq p_x \\ &\simeq p_y \end{aligned}\right.$$

These are the forms in LICs around $\theta \simeq 0$